Optimization of Smith-Purcell radiation at very high energies

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A theoretical analysis of Smith-Purcell radiation at very high energies is presented. The energy per unit frequency and solid angle is expressed in closed form as a function of the grating geometry, beam energy, and viewing angles. A certain choice of grating geometry is shown to optimize the output energy for a particular order of radiation. Scaling laws are derived for the energy emitted into all orders of radiation in the relativistic limit. It is shown that the total energy emitted into each order scales as the three-halves power of the beam voltage.

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I. INTRODUCTION

The interaction of an electron beam with a metal grating and the emission of radiation was first reported by Smith and Purcell in 1953 [1] and continues to attract interest because it offers the possibility of a compact and efficient method of generating a high-power, tunable source of radiation, particularly in the far infrared. Emphasis in this communication is on the high-energy limit of the Smith-Purcell (SP) mechanism. In particular the conditions required for strong peaking of the intensity in the beam direction will be identified.

Previous theoretical work on SP radiation falls roughly into one of two categories. The first utilizes a plane-wave diffraction formalism suggested by Toraldo di Francia [2] and developed by van den Berg [3]. A number of variations on this theme can be found in the literature. An analysis of finite conductivity strip gratings is provided by Petit and Tayeb [4]. Others have modeled the scattering of plane waves from arrays of conducting bars, including Rubin and Bertoni [5] and Shiao and Peng [6]. In this approach the electromagnetic fields are expanded in terms of Fourier integrals and infinite matrix equations containing the amplitude of excitation of various plane waves are truncated and solved in approximate fashion. At small angles this scheme breaks down. This difficulty arises because an increasingly large number of terms must be incorporated as the angle is decreased to maintain convergence and numerical instability results. This is a well known limitation of the theory and one that is recognized by its practitioners. ''We have restricted our calculations to observation angles [$\theta > 45°$] where reliable convergence of the numerical solution of the integral equations was obtained'' [7].

The second approach is based on the approximation to diffraction theory employing the Huygens-Kirchhoff integrals. Chapter 9 of Jackson's textbook on electrodynamics contains a discussion of these techniques [8]. This work originated nearly at the same time as that of di Francia. In particular, two authors, Bolotovskii and Voskresenskii, analyzed the SP radiation produced by a charged particle passing over a perfectly conducting strip grating using the scalar Kirchhoff theory [9]. The result of their calculation [Eq. (34) of [9]] for the spectral intensity of the SP radiation agrees precisely with the result found in the work of Brownell *et al.* [Eq. (11) of [11]]. This result is analyzed here to determine the experimental parameters that produce maximum SP emission. It is rederived here for the convenience of the reader with an emphasis on the physical assumptions of the theory and the intuitive nature of the model.

Forward directed emission of SP radiation was first verified in experiments performed using a 2.8 MeV/c electron beam interacting with a 1 cm period grating at the Brookhaven National Laboratory [10].

We have recently developed a theory describing SP radiation by considering the surface charge induced on the grating by the charge passing over the grating and ''dragged'' along with it. This approach is described in detail in [11] and has the advantage of being physically intuitive and mathematically tractable. It will be shown here that the peak of SP radiation is expected for precisely the angular range excluded by diffraction theory analysis and that higher-energy beams are capable of producing more narrowly focused and greater energy SP output than low-energy beams.

The image current theory provides a formula for the spectral distribution of the emitted energy in terms of the beam and grating parameters. Figure 1 illustrates the properties of the grating under consideration. A strip grating is considered rather than an echelle or lamellar grating, for example, to eliminate the added complications of varying the depth of grooves in the grating. It will be shown that the broad features of the results of the analysis are changed only slightly due to the more complicated diffraction efficiency of other grating types. The variables defining our case are as follows. The charge of the particle passing over the grating is q. The speed of the particle is βc. The impact parameter of the charge with respect to the grating is b. The angles θ and ϕ locate the direction toward an observer in the far field with respect to the beam direction. The grating has total length L and period l, and is composed of strips having width s. The subsequent analysis demonstrates an optimal choice of parameters to generate SP radiation and scaling laws for the angular distribution of the radiation and the energy generated with beam energy.

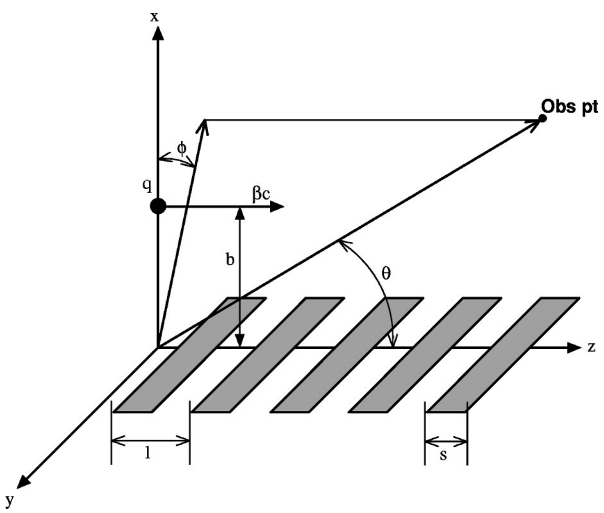

FIG. 1. Strip grating geometry showing the grating period l, strip width s, impact parameter b, and charge q moving at speed βc. The observer is oriented using the standard spherical coordinates θ and ϕ relative to the z axis.

II. DERIVATION OF INTENSITY PROFILE

The electromagnetic waves radiated by a system of currents carry energy into different directions. The total energy per unit frequency and solid angle is related to the expression for current density by the following [8]:

$$\frac{d^2 I}{d\omega\, d\Omega}=\frac{\omega^2}{4\pi^2 c^3}\left|\int\int dt\, d\vec{x}\hat{u}\times[\hat{u}\times\vec{J}(\vec{x},t)]e^{i\omega[t-\hat{u}\cdot\vec{x}/c]}\right|^2 \tag{1}$$

where $\hat{u}=\hat{x}\sin\theta\cos\phi+\hat{y}\sin\theta\sin\phi+\hat{z}\cos\theta$ is the unit vector from the point of integration $\vec{x}$ to the point of observation and the integration is performed over all time and space. This expression should be considered a function of $\hat{u}$ and frequency. The strip grating is periodic in z and the currents are generated by the source charge which travels with velocity $\beta c\hat{z}$. The distribution of image charge on the strip grating is

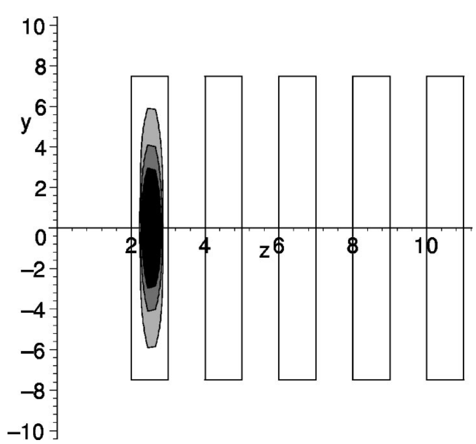

FIG. 2. Illustration showing the distribution of image charge on the strip grating for high energy where the ''footprint'' is confined to a single strip. The three contours indicate the locations where the charge density is equal to 20%, 10%, and 0.4% of the maximum value. In this example, $\gamma=20.0$. A high-energy beam ($\gamma>100$) induces an extremely narrow footprint.

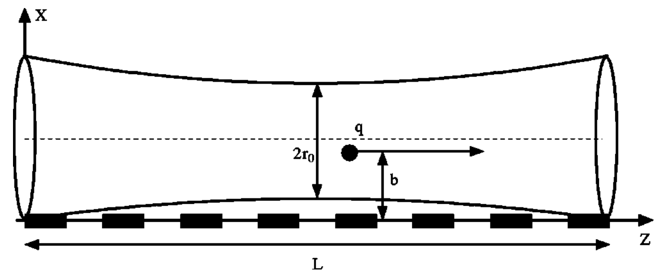

FIG. 3. Strip grating geometry showing the relationship between the length of the grating L, the waist diameter of the beam $2r_0$, and the impact parameter b.

shown schematically in Fig. 2. Therefore, we can describe the total current as the sum over the current in each strip,

$$\vec{J}(\vec{x},t)=\sum_{m=1}^{N}\vec{J}_{tooth}\left(\vec{x}-ml\hat{z},t-\frac{ml}{\beta c}\right). \tag{2}$$

The general form of the Smith-Purcell radiation spectrum can be derived without specifying the exact expression for $\vec{J}_{tooth}$. Interchange the order of summation and integration in Eq. (1) and change the coordinates of integration for each term in the sum using $\vec{x}'=\vec{x}-ml\hat{z}$ and $t'=t-ml/(\beta c)$. It is found that

$$\frac{d^2 I}{d\omega\, d\Omega}=\frac{\omega^2}{4\pi^2 c^3}\left|\sum_{m=1}^{N} e^{i\omega[ml/(\beta c)-\hat{u}\cdot\hat{z}ml/c]}\vec{\mathcal{J}}\right|^2, \tag{3}$$

where

$$\vec{\mathcal{J}}=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dt\hat{u}\times[\hat{u}\times\vec{J}_{tooth}(\vec{x},t)]e^{i\omega[t-\hat{u}\cdot\vec{x}/c]}, \tag{4}$$

and the primes have been dropped immediately after making the substitution. Since $|z_1 z_2|^2=|z_1|^2|z_2|^2$ we have

$$\frac{d^2 I}{d\omega\, d\Omega}=\frac{\omega^2}{4\pi^2 c^3}|\vec{\mathcal{J}}|^2\mathcal{K}, \tag{5}$$

where

$$\mathcal{K}=\left|\sum_{m=1}^{N} r^m\right|^2, \tag{6}$$

a geometric series, and $r=\exp[(i\omega l/c)(1/\beta-\cos\theta)]$. The series has the well known closed form expression,

$$\mathcal{K}=\frac{\sin^2[(\omega Nl/2c)(1/\beta-\cos\theta)]}{\sin^2[(\omega l/2c)(1/\beta-\cos\theta)]}. \tag{7}$$

Equation (7) contains the frequency envelope of the SP radiation. The full width at half maximum output in frequency can be expressed as

$$\Delta\omega=\frac{1}{N}\frac{4c}{l}\left(\frac{1}{\beta}-\cos\theta\right)^{-1}. \tag{8}$$

If the grating has many periods then $\mathcal{K}$ sharpens into a series of δ functions:

$$\mathcal{K} \xrightarrow{N\gg 1} \sum_{n\neq 0} \frac{\omega L}{|n| l}\delta(\omega-\omega_n). \tag{9}$$

This forces the spectrum to resolve into a set of discrete lines according to the ''Smith-Purcell relationship'':

$$\omega_n \equiv \frac{2\pi|n|c\beta}{l(1-\beta\cos\theta)}. \tag{10}$$

We can integrate the energy per frequency per solid angle over frequency to yield the total energy per solid angle:

$$\frac{dI}{d\Omega} = \int_0^\infty \frac{\omega^2}{4\pi^2 c^3}|\vec{\mathcal{J}}|^2 \mathcal{K} d\omega. \tag{11}$$

The δ function form of the frequency spectrum allows this integral to be performed by inspection. At this point, we can incorporate the precise form of the image current derived in the Appendix. Using the Smith-Purcell relationship to eliminate ω_n and incorporating the expression for the ''transformed'' image current $|\vec{\mathcal{J}}|^2$, we have

$$\frac{dI}{d\Omega} = \frac{2q^2 N}{\pi l}\frac{\beta^3 \sin^2\theta}{(1-\beta\cos\theta)^3} \times \sum_{n\neq 0} \exp\left(-\frac{4\pi|n|b\sqrt{1+(\gamma\beta\sin\theta\sin\phi)^2}}{\gamma l(1-\beta\cos\theta)}\right) G, \tag{12}$$

where the dependence of the radiation on the particular choice of grating has been separated out by defining $G \equiv \sin^2(\pi|n|s/l)$. For lamellar or sinusoidal gratings, for example, the above expression for $dI/d\Omega$ would be altered only in the form of G. This result is a special case of the more general theory derived in Eq. (11) of [11]. The energy distribution function is similar to those derived for other radiative processes such as transition radiation [12] and Cerenkov radiation [8].

The polarization of the SP radiation varies as a function of observation angle. An observer looking along the direction (θ,ϕ) sees radiation with wave vector parallel to $\hat{u}$. The magnetic field is directed along $\hat{B}=\hat{z}\times\hat{u}$. The electric field is polarized in the plane of $\hat{u}$ and $\hat{z}$. For example, radiation directed along $\theta=90°$ and $\phi=0°$ has $\hat{E}=-\hat{z}$, $\hat{B}=\hat{y}$, and $\hat{k}=\hat{x}$.

III. OPTIMIZATION OF RADIATED ENERGY

The energy emitted per solid angle is a complicated function of seven variables. The question arises, then, of how to optimize the SP radiation by judicious choice of grating geometry. Intuitively, a number of features can be noted. Because this model does not include the effects of the radiative fields back on the source charge, the charge is not accelerated. Therefore, a longer grating will result in greater output. However, there is a practical limit to the useful length of grating due to the nonzero divergence angle of any beam in the absence of focusing fields. The exponential damping term in the expression for the emitted energy requires, independent of all other considerations, that $\phi=0$ to maximize output. The remaining terms must be chosen to make the argument of the exponential small. Therefore, the impact parameter b will be chosen to bring the beam as close to the grating as possible. The energy of the beam, and hence γ, should be large. However, the angular expression in the form factor, $\sin^2\theta/(1-\beta\cos\theta)^3$, will force the maximum of the radiation at high energy to small angles. It will be shown that the interplay between these two effects results in a simple analytic expression for the global optimum.

With a rough conceptual understanding of the issues involved in optimizing the SP radiation, a rigorous analytical derivation can be performed. Let us assume the following: The total length of the grating, L, is chosen so that a beam that has radius r_0 at the middle of the grating expands enough to begin to intercept the grating at the leading edge due to a finite beam emittance ϵ. In a drift space, an emittance dominated beam expands according to [13]

$$r(z)^2 = r_0^2 + (z-z_0)^2\epsilon^2/r_0^2. \tag{13}$$

We assume that the beam is placed close to the grating to maximize the output, $b=r_0\sqrt{2}$. Figure 3 illustrates the relationship of emmitance for the length of the grating and the impact parameter. In practice b will be a multiple of r_0 of order unity that will depend on the precise beam density profile and the tolerance of the grating for intercepting high-energy electrons. For the present discussion we simply set this factor for convenience without significant loss of generality. We find that $L=2r_0^2/\epsilon$. The normalized emittance $\epsilon_N \equiv \epsilon\gamma\beta$ characterizes the electron source. And so, alternatively,

$$L = b^2\gamma\beta/\epsilon_N. \tag{14}$$

For convenience, introduce $x\equiv\cos\theta$ and break up the series into terms as follows:

$$\frac{dI}{d\Omega} = \sum_{n\neq 0}\frac{dI_n}{d\Omega} = \frac{q^2}{\epsilon_N}\sum_{n\neq 0} f_n(x,\phi). \tag{15}$$

The total energy per solid angle emitted from the strip grating is the sum of energy emitted in various terms as described by Eq. (12). How may these contributions be denoted? In traditional microwave tube nomenclature a particular ''mode'' of output corresponds to a particular frequency and spatial output pattern, e.g., the TE_{01} rectangular waveguide mode at 17.14 GHz. In the case of SP radiation, where each angle of observation corresponds to a different frequency of emission, some care needs to be taken with labels. Following diffraction grating nomenclature, each term in the summation will be referred to as a particular ''order'' of SP radiation. Each order consists of energy emitted into different wavelengths, and therefore angles, in accordance with the Smith-Purcell relationship. The functional dependence of the nth order on beam energy, grating geometry, and observation direction is contained within the ''form factor'':

$$f_n(x,\phi)=\frac{2b^2\gamma\beta^4}{\pi l^2}\frac{(1-x^2)}{(1-\beta x)^3}\times\exp\left(-\frac{4\pi|n|b\sqrt{1+\gamma^2\beta^2(1-x^2)\sin^2\phi}}{\gamma l(1-\beta x)}\right)\times\sin^2\left(\frac{\pi|n|s}{l}\right). \tag{16}$$

IV. OPTIMIZATION OF A SELECTED ORDER

Assume that a particular order $\overline{n}$ has been chosen to be optimized. A set of parameters $\{\phi_{\overline{n}},\theta_{\overline{n}},s_{\overline{n}},l_{\overline{n}}\}$ may be found that will maximize the form factor $f_{\overline{n}}$. The angle ϕ appears only in the exponent and, clearly, $\phi=0$ will maximize the output. Similarly, the strip width s appears only in the $\sin^2(\pi|\overline{n}|s/l)$ term and has an optimal value $s_{\overline{n}}=lp/(2|\overline{n}|)$ where p is an odd integer. According to the definition of the strip width s, we must have $s\leqslant l$. Therefore, $p\leqslant 2|\overline{n}|$. For example, $|\overline{n}|=3$ implies that $p\in\{1,3,5\}$. Equation (16) evaluated for $n=\overline{n}$ reduces to

$$f_{\overline{n}}(x,\phi=0)=\frac{2b^2\gamma\beta^4}{\pi l^2}\frac{(1-x^2)}{(1-\beta x)^3}\exp\left(-\frac{4\pi|\overline{n}|b}{\gamma l(1-\beta x)}\right). \tag{17}$$

The grating period must be chosen to maximize the output energy. Set the derivative of the form factor with respect to l to zero, $\partial f_{\overline{n}}/\partial l\equiv 0$. This yields $l=2\pi b|\overline{n}|/[\gamma(1-\beta x)]$. The optimum period depends on the emission angle of interest. Substituting this expression for l into $f_{\overline{n}}$ we have

$$f_{\overline{n}}=\frac{e^{-2}}{2\pi^3}\frac{\gamma^3\beta^4}{|\overline{n}|^2}\frac{(1-x^2)}{(1-\beta x)}. \tag{18}$$

Take the derivative of $f_{\overline{n}}$ with respect to x to optimize over θ and solve the resulting quadratic equation. The two solutions are $x_{\overline{n},\pm}=\beta^{-1}\pm\sqrt{\beta^{-2}-1}$ but the only acceptable solution has an absolute value less than or equal to 1 due to the definition of x:

$$\theta_{\overline{n}}=\cos^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\right)\overset{\gamma\gg 1}{\rightarrow}\sqrt{\frac{2}{\gamma}}. \tag{19}$$

It should be noted that, for extremely high energies, the angle of the peak output is close to zero and becomes difficult to separate from the beam unless a bending magnet is utilized downstream. A blazed grating can be used to tilt the angle of the peak radiation back slightly. Blazed gratings will be treated in future publications. Substitution of this angle into the expression for the optimum period fixes the grating geometry:

$$l_{\overline{n}}=2\pi b|\overline{n}|\quad\text{and}\quad s_{\overline{n}}=p\pi b. \tag{20}$$

At this point in the analysis we can ''close the circle'' by using Eqs. (19) and (20) in the previous expressions which were functions of these variables. By virtue of the Smith-Purcell relationship between angle and frequency, we can compute the wavelength of radiation corresponding to the peak output at $\theta_{\overline{n}}$,

$$\lambda_{\overline{n}}=2\pi b/(\gamma\beta). \tag{21}$$

The width of the surface current footprint is b/γ, implying a spectral bandwidth of approximately γ/b. Therefore, this result also represents the minimum wavelength that this device can generate.

A self-consistency check can be made for the assumption that the number of grating periods is large, $N\gg 1$. Because the grating length L is simply N times the period of the grating, $l_{\overline{n}}=2\pi b|\overline{n}|$, we find that

$$N_{\overline{n}}=\frac{L}{l}=\frac{b}{\epsilon_N}\frac{\gamma\beta}{2\pi|\overline{n}|}, \tag{22}$$

where the subscript reflects the dependence of the total number of periods in the optimized grating on the choice of order $\overline{n}$. For a typical high-brightness beam with normalized emittance on the order of 1π mm mrad and beam size (and therefore impact parameter) on the order of 10 μm, we find that $N\approx\gamma\gg 1$.

V. OTHER ORDERS

The strip grating geometry has been chosen to maximize the production of a particular order of SP radiation, $\overline{n}$. However, radiation will be produced for all orders $n\leqslant -1$. What are the values of θ and ϕ that maximize the output for orders other than $\overline{n}$? Inserting the optimized grating period and strip width, the expression for $f_n(x,\phi)$ becomes

$$f_n(x,\phi)=\frac{\gamma\beta^4}{2\pi^3|\overline{n}|^2}\frac{(1-x^2)}{(1-\beta x)^3}\times\exp\left[-\frac{2n\sqrt{1+\gamma^2\beta^2(1-x^2)\sin^2\phi}}{\overline{n}\gamma(1-\beta x)}\right]\times\sin^2\left(\frac{\pi}{2}\frac{n}{\overline{n}}p\right). \tag{23}$$

As before we immediately note that $\phi=0$. The angle θ can be determined, using $\partial f_n/\partial x\equiv 0$. This yields a cubic equation in x,

$$\left(3-\frac{2}{\gamma}\frac{n}{\overline{n}}\right)\beta-(2+3\beta^2)x+\left(1+\frac{2}{\gamma}\frac{n}{\overline{n}}\right)\beta x^2+\beta^2x^3=0. \tag{24}$$

For the strongly relativistic case, $\gamma\gg n/\overline{n}$ we find the asymptotic solution

$$\theta_n\approx\theta_{\overline{n}}\sqrt{\frac{n}{\overline{n}}}. \tag{25}$$

The maximum value of $f_n(x_n,\phi=0)$ in this limit is approximately

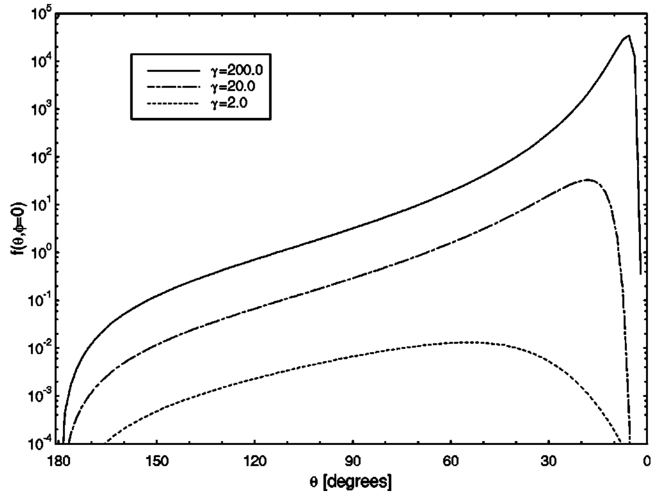

FIG. 4. A plot of the form factor at constant $\phi=0$ for θ from 0 to 180° for the case of $n=\bar{n}=1$, $p=1$, and $\gamma=2.0$, 20.0, and 200.0.

$$f_n(x_n,\phi=0)\approx\frac{e^{-2}}{2\pi^3}\beta^2\gamma^2(\gamma-1)\sin^2\left(\frac{\pi}{2}\frac{n}{\bar{n}}p\right). \quad (26)$$

These expressions become exact for all values of γ when $n=\bar{n}$. The angle at which the nth order is maximized can be substituted back into the Smith-Purcell relationship to derive the angular frequency of the SP radiation for each order $\omega_n=c\gamma/b$ in the high-energy limit. Furthermore, knowing the number of periods in the grating, we are in a position to deduce the width in frequency of each order using the expression in Eq. (8):

$$\Delta\omega_n=4\frac{\bar{n}}{n}\frac{c\epsilon_N}{b^2}. \quad (27)$$

VI. ANGULAR DISTRIBUTION OF RADIATION

The next question concerns the angular distribution of the radiation. We expect, crudely, that at high energy, $\gamma\gg1$, the radiation of each order will become a narrow cone in θ and ϕ. What is the total energy emitted in each order? Does this energy increase with beam energy? Figures 4 and 5 show the variation of the form factor with θ and ϕ for beams of several energies. The energy is maximized at smaller angles for higher energy as expected from Eq. (25). As expected, the energy is also maximized at $\phi=0$.

Introduce the following variables to denote the positions of the half-output points. The implicit definition of the half-output point $\phi_{n,1/2}$ is $f_n(x_n,\phi_{n,1/2})=\frac{1}{2}f_n(x_n,0)$. This equation can be solved analytically. Using the small angle approximation, appropriate in this case,

$$\phi_{n,1/2}\approx0.64\gamma^{-1/2}\sqrt{\frac{\bar{n}}{n}}. \quad (28)$$

Twice this value yields $\Delta\phi_n$. The two half-output points along θ are defined similarly by $f_n(x_n^{\pm},\phi=0)=\frac{1}{2}f_n(x_n,\phi=0)$ where $x_n^-\in[0,x_n]$ and $x_n^+\in[x_n,\pi]$. The full width at half maximum (FWHM) is $\Delta\theta_n=[\cos^{-1}(x_n^+)-\cos^{-1}(x_n^-)]$. We find that

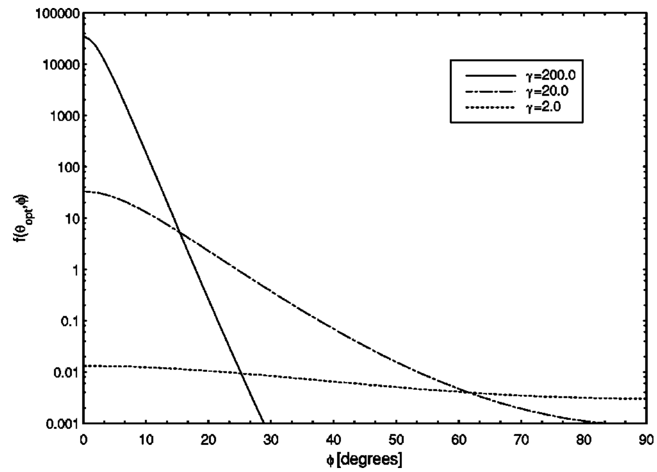

FIG. 5. A plot of the form factor at constant θ (that yielding the peak output) for ϕ from 0 to 90° for the case of $n=\bar{n}=1$, $p=1$, and $\gamma=2.0$, 20.0, and 200.0.

$$\Delta\theta_n\approx1.3\gamma^{-1/2}\sqrt{\frac{n}{\bar{n}}} \quad (29)$$

The solid angle subtended by each lobe is $\Delta\Omega_n\approx\sin\theta_n\Delta\theta_n\Delta\phi_n$,

$$\Delta\Omega_n\approx2.4\gamma^{-3/2}\sqrt{\frac{n}{\bar{n}}} \quad (30)$$

The total energy in each lobe is $\varepsilon_n\approx(q^2/\epsilon_N)f_n(x_n,\phi=0)\Delta\Omega_n$ or

$$\varepsilon_n\approx0.01\gamma^{3/2}\frac{q^2}{\epsilon_N n^2}\sin^2\left(\frac{\pi}{2}\frac{n}{\bar{n}}p\right)\sqrt{\frac{n}{\bar{n}}} \quad (31)$$

Evidently, the radiant energy increases with the 3/2 power of the beam energy. The variation in energy with order is such that the maximum output is obtained for the case of $n=\bar{n}=p=1$. The energy oscillates with increasing order but is bounded above by $n^{-3/2}$. Depending on the choice of $\bar{n}$ and p, an odd integer less than $2|\bar{n}|$, the magnitudes of the energies in the various orders will be altered accordingly.

As with any periodic radiator, the minimum resolution is $\Delta\lambda/\lambda\approx1/N$. Because of the SP dispersion relationship the angular width of the radiated energy lobe described by Eq. (29) corresponds to a spectral width of $\Delta\lambda/\lambda\approx1.8\gamma^{-2}$. In practice the larger of these two linewidths will dominate depending on the particular parameters of the beam source being utilized.

The average power produced by a continuous current can be approximated by recognizing that the radiation produced by individual charges sums incoherently when the plasma wavelength of the beam is much larger than the wavelength of the Smith-Purcell radiation. In this case the net energy production scales linearly with the number of radiators (as

TABLE I. SP radiation from existing or planned accelerators.

	Facility					
	Duke[a] [14]	TUM[b] [15]	CIRFEL[c] [16]	SDALINAC[d] [17]	BNL-ATF[e] [18]	LCLS[f] [19]
Energy (MeV)	1.2	3	14	38	50	14350
ϵ_N (π mm mrad)	3.5	5	20	2	1	1.5
$r_0=b/\sqrt{2}$ (μm)	250	100	350	50	25	0.5
σ_z (mm/ps)	0.3/1	3000/10^4	3.0/10	1.2/4	0.6/2	0.02/0.067
I (A)	20	0.5	200	1.5	50	3400
l (mm)	2.2	0.89	3.1	0.44	0.22	0.009
s (mm)	1.1	0.44	1.6	0.22	0.11	0.004
N	51	30	110	420	560	4200
λ (μm)	700	130	110	5.9	2.3	0.00032
$\Delta\lambda/\lambda$	0.16	0.038	0.009	0.002	0.002	0.0002
θ (deg)	43	30	15.1	9.3	8.1	0.48
$\Delta\Omega$ (mSr)	390	130	16	4	2	5×10^{-4}
P (mW)	0.5	0.026	22	7.0	710	1.5×10^{8}

[a]The Mark III Free Electron Laser Linear Accelerator Driver.
[b]3 MV Van de Graaff accelerator at the Physics Department of the Technical University, Munich.
[c]The compact infrared free electron laser.
[d]The Superconducting Darmstadt Linear Accelerator.
[e]The Brookhaven National Laboratory Accelerator Test Facility.
[f]The Linac Coherent Light Source.

opposed to quadratically as would be the case with coherent radiation from a single macrocharge). The power produced is the product of the rate at which charges pass the grating and the energy produced by each charge, $P_n=I\varepsilon_n/q$.

It should be noted that the output power does not scale with the canonical figure of merit, the ''normalized beam brightness'' $B_N=I/(\pi\epsilon_n)^2$. Instead, the power is proportional to current divided by normalized emittance to the first power. This relationship is a consequence of the geometry of the Smith-Purcell effect. The beam-grating coupling drops exponentially with the impact parameter of the charges but a transverse displacement has no effect. By Liouville's theorem, the total phase space area subtended by the beam is conserved. However, the components of the normalized emittance, ϵ_{Nx} and ϵ_{Ny}, are not conserved individually. Therefore, it is possible to ''squeeze'' a round beam into an elliptical cross section in order to increase the vertical beam brightness at the expense of the transverse brightness. Indeed, the Smith-Purcell effect is naturally suited to a sheet beam geometry.

To provide a quantitative sense of these results, Table I lists the operating parameters of a number of existing or planned accelerators. The parameters of each electron beam are the beam kinetic energy, normalized emittance, radius, bunch length, and peak current. The calculated quantities are the grating period and strip width, the number of periods in the grating, the peak wavelength of the output, the angle of maximum emission, the solid angle subtended by the main lobe of radiation, and the peak power produced.

VII. CONCLUSIONS

A theoretical analysis of Smith-Purcell radiation from a perfectly conducting strip grating has been presented. The energy per unit frequency and solid angle was derived. A certain choice of grating geometry was shown to optimize the output energy for a particular order of radiation. In the highly relativistic limit, it has been shown that the energy emitted in each order scales as the 3/2 power of the beam energy. It has also been shown that the emitted energy is focused into a narrow cone at high energy, emerging at an angle close to the direction of the beam. The optimal parameters for SP radiation are summarized in Table II.

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TABLE II. Summary of SP optimization for a given order $\bar{n}$.

Grating	$l_{\bar{n}}=2\pi b\lvert\bar{n}\rvert$	$s_{\bar{n}}=p\pi b$[a]	$N_{\bar{n}}=b\gamma\beta/(2\pi\epsilon_N\lvert\bar{n}\rvert)$
Angles	$\theta_n=\cos^{-1}[\sqrt{(\gamma-1)/(\gamma+1)}\sqrt{n/\bar{n}}$	$\phi_n=0$	$\Delta\Omega_n\approx2.4\gamma^{-3/2}\sqrt{n/\bar{n}}$
Output	$\omega_n=c\gamma\beta/b$	$\lambda_n=2\pi b/(\gamma\beta)$	$\Delta\omega_n=4c\epsilon_N(\bar{n}/n)/b^2$
Energy	$\varepsilon_n\approx0.01\gamma^{3/2}(q^2/\epsilon_N n^2)\sin^2[(\pi/2)(n/\bar{n})p]\sqrt{n/\bar{n}}$		
Power		$P_n=I\varepsilon_n/q$	

[a]p is an odd integer such that $p\leqslant2\lvert\bar{n}\rvert$.

APPENDIX: DERIVATION OF THE TRANSFORMED IMAGE CURRENT

The induced surface current on the grating is computed by first considering a charge q at a distance b above a perfectly conducting plane. A surface charge develops that is described by

$$\sigma(y,z) = -\frac{1}{4\pi}\frac{2qb}{(b^2+y^2+z^2)^{3/2}}. \tag{A1}$$

If the charge trajectory is $\vec{r}(t) = b\hat{x} + \beta ct\hat{z}$ and the conductivity of the metal is infinite, then the image charge moves accordingly. The standard relativistic substitutions are used: $t' = \gamma(t-\beta cz)$ and $z' = \gamma(z-\beta ct)$. Also, since σ is the charge per unit area and the dimension of length along $\hat{z}$ is Lorentz contracted, $\sigma \rightarrow \gamma\sigma$. A Dirac δ function describes the confinement of the current to the surface, $x=0$:

$$\vec{J}(x,y,z,t) = -(\beta c\hat{z})\frac{\delta(x)2q\gamma b/(4\pi)}{[b^2+y^2+\gamma^2(z-\beta ct)^2]^{3/2}}. \tag{A2}$$

For the problem at hand, the charge is assumed to travel over a grating composed of strips of width s arranged with periodicity l. It is assumed that the image current in this situation is that calculated for the uninterrupted metal plane considered above where metal is present in strips and zero in the gaps between the metal regions. This is accomplished by defining a function $g_{tot}(z)$ that is zero in the gaps between strips and unity on the strips. Using the unit Heaviside step function $\Theta(z)$, define

$$g(z;a,b) = \Theta(z-a) - \Theta(z-b) = \begin{cases} 1 & \text{if} \quad a<z<b \\ 0 & \text{if} \quad z<a \quad \text{or} \quad z>b. \end{cases} \tag{A3}$$

The grating has N strips and the individual strips are indexed by m which runs from 1 to N. The grating is described by $g_{\rm tot}(z) = \Sigma_{m=1}^{N} g(z;ml,ml+s)$. The final expression for the system of source currents due to the passage of the charge over the strip grating is

$$\vec{J}(x,y,z,t) = -g_{tot}(z)(\beta c\hat{z})\frac{\delta(x)2q\gamma b/(4\pi)}{[b^2+y^2+\gamma^2(z-\beta ct)^2]^{3/2}}. \tag{A4}$$

Figure 2 illustrates the shape of the induced surface currents. By comparison with the definition for the current density contribution from each tooth, we have

$$\vec{J}_{tooth}(\vec{x},t) = -(\beta c\hat{z})g(z;0,s)\frac{\delta(x)2q\gamma b/(4\pi)}{[b^2+y^2+\gamma^2(z-\beta ct)^2]^{3/2}}. \tag{A5}$$

This result permits the computation of the "transformed" image current $\vec{\mathcal{J}}$ defined in Eq. (4). The cross products factor out of $\vec{\mathcal{J}}$ and reduce to $|\hat{u}\times\hat{u}\times\hat{z}|^2 = |\hat{u}\times\hat{z}|^2 = \sin^2\theta$. After performing the trivial x integration, and substituting $\tau = \gamma(\beta ct - z)$,

$$\vec{\mathcal{J}} = \hat{u}\frac{qb}{2\pi}\sin\theta\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}d\tau\frac{g(z;0,s)}{(b^2+y^2+\tau^2)^{3/2}} \times\exp\left[\frac{i\omega}{c}\left(\frac{\tau}{\gamma\beta}+\frac{z}{\beta}+y\sin\theta\sin\phi+z\cos\theta\right)\right], \tag{A6}$$

neglecting an overall complex phase which has no effect in the calculation of $|\vec{\mathcal{J}}|^2$. This relatively complicated expression simplifies rapidly upon recognizing the integration over y as that of a modified Bessel function of the second kind [20]:

$$\vec{\mathcal{J}} = \hat{u}\frac{\sin^2\theta\sin\phi qb\omega}{\pi c}\int_0^s dz\exp\left(\frac{i\omega z}{\beta c}(1-\beta\cos\theta)\right) \times\int_{-\infty}^{\infty}d\tau\frac{K_1((\omega/c)\sin\theta\sin\phi\sqrt{b^2+\tau^2})}{\sqrt{b^2+\tau^2}}\exp\left(-\frac{i\omega b\tau}{\gamma\beta c}\right). \tag{A7}$$

The remaining integrations over z and τ separate, the former yielding a simple exponential over the interval $[0,s]$ and the latter another modified Bessel function which reduces to a simple exponential function, $K_{1/2}(z) = e^{-z}\sqrt{\pi/(2z)}$. At the end of the calculations we find

$$|\vec{\mathcal{J}}|^2 = \frac{4q^2\beta^2c^2}{\omega^2}\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}\sin^2\left(\frac{\omega s(1-\beta\cos\theta)}{2\beta c}\right) \times\exp\left(-\frac{2b}{\lambda_e}\right), \tag{A8}$$

where the evanescent length scale is defined by $\lambda_e^{-1} \equiv (\omega/\gamma\beta c)\sqrt{1+(\gamma\beta\sin\theta\sin\phi)^2}$. Physically, the evanescent length scale comes from $k_x = i\lambda_e^{-1}$, the component of the wave vector perpendicular to the grating.

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